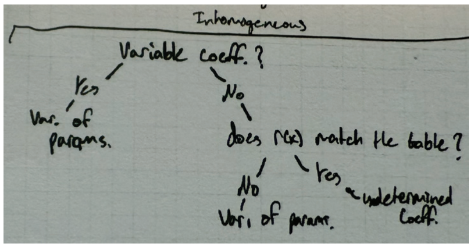
My Notes on Solving ODEs

Linear 1st-Order ODEs

$y' + py = r$

- homogeneous: $y_h = C_0 e^{-\int p dx}$
- inhomogeneous: $y = e^{-\int p dx}(\int e^{\int p dx} r dx + C)$ derived from integrating factor $\mu = e^{\int p dx}$



Non-Linear 1st-Order ODEs

Bernoulli equation for $y' + py = qy^n$:

1. let $u = y^{1-n}$
2. Transform into $u' + p(1-n)u = q(1-n)$ to use integrating factor
   (a) You can isolate y in terms of u and find y' in terms of u and u' to turn everything in terms of u

System of linear 1st-order ODEs (chap 5)

- $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ ansatz $y = \eta e^{\lambda t}$ like $y = e^{\lambda t}$ for $y' = \lambda y$ but w/ vector coeff

$\vec{y}' = A\vec{y}$

1. find A's eigenvalues λ_1 and λ_2 from A's characteristic equation $|A - \lambda I| = 0$ or $\lambda^2 - tr(A) + \det A = 0$
2. find $\eta_{\{1,2\}}$ from $\lambda_{\{1,2\}}$ where $(A - \lambda_1 I)\eta_1 = 0$
3. solution is $\vec{y} = c_1 \vec{\eta}_{\lambda_1} e^{\lambda_1 t} + c_2 \vec{\eta}_{\lambda_2} e^{\lambda_2 t}$ if $\lambda_1 \neq \lambda_2$

   from initial conditions

   $\vec{y} = c_1 \vec{\eta} e^{\lambda t} + c_2 \vec{\eta} t e^{\lambda t} + c_2 \vec{p} e^{\lambda t}$ if only one eigenvalue

   where generalized eigenvector $\vec{p}$ is in $(A - \lambda I)\vec{p} = \vec{\eta}$

   $\vec{y} = c_1 e^{\alpha t}(\vec{\eta}_R \cos\beta t - \vec{\eta}_I \sin\beta t) + c_2 e^{\alpha t}(\vec{\eta}_I \cos\beta t + \vec{\eta}_R \sin(\beta t))$

   if $\lambda_1 = \alpha + \beta i$, $\vec{\eta}_1 = \vec{\eta}_R + \vec{\eta}_I i$

   $\lambda_2 = \alpha - \beta i$, $\vec{\eta}_2 = \vec{\eta}_R - \vec{\eta}_I i$

- Turn higher-order ODEs into system of 1st-order ODEs with $z = y'$

My Notes on Other Subjects

BVP

auxiliary conditions = order of ODE

Finite approximation of derivatives; from Taylor series expansion

forward diff $y_j' = \frac{y_{j+1} - y_j}{h}$

central diff $y_j' = \frac{y_{j+1} - y_{j-1}}{2h}$

$y_j'' = \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} \because = \frac{\frac{y_{j+1} - y_j}{h} - \frac{y_j - y_{j-1}}{h}}{h}$

direct method discretizes t and represents all derivatives of y w/ $y_{n\pm 1}$ so $A\vec{y} = \vec{f}$ where A is tri-diagonal matrix. Solving for y to get value at each index: y_j.

fixed y_L and for y_R:

(fixed-value) Dirichlet boundary conditions – know y_R ($y_2 \cdots y_{N-1}$ unknown)

fixed (derivative) Neumann " – know y_R'

(mixed condition) Robin " – know eqn relating y_R and y_R'

- f has equilibrium at $y*$ if the ODE is autonomous (meaning $f = \frac{dy}{dt}$ is the same regardless of t) and y-val for $f(y*) = 0$
  - stable vs. unstable can be determined by using table of factors or evaluating $f'(y*)$'s sign
  - Semi-stable means stable in one direction and unstable on the other
- separation of variables: $\frac{dy}{dx} = f(x)g(y) \to \int \frac{dy}{g} = \int f dx + C$
- reduction for separation of variables:
  - if $y' = f(\frac{y}{x}), u = \frac{y}{x}$ and $y' = u'x + u \quad (\because y = ux)$
  - if $y' = f(ax + by + c)$, $u = ax + by + c$ and find y' in terms of u and u'

Existence and Uniqueness

- For $y' = f(x, y)$ and $y(x_0) = y_0$ meaning point is (x_0, y_0)
- Theorem 1 (existence theorem) - if $f(x, y)$ continuous at all points in R_1 containing point (x_0, y_0) then f has 1+ solns $y(x)$ passing through the point
- Theorem 2 (uniqueness theorem) - if $f(x, y)$ and $f_y(x, y)$ (AKA $\frac{\partial f}{\partial y}$) continuous in R_2 containing point, then f has a **unique** solution passing through the point
- Notes
  - R_1 is region containing point where f continuous
  - R_2 is region containing point where f_y continuous
  - $R_1 \cap R_2$ is region of validity and x-range is interval of validity

Numeral Methods

(forward) Euler: $y_{n+1} = y_n + hy_n'$ is explicit $\because$ all RHS terms known

backward Euler: $y_{n+1} = y_n + hy_{n+1}'$ is implicit

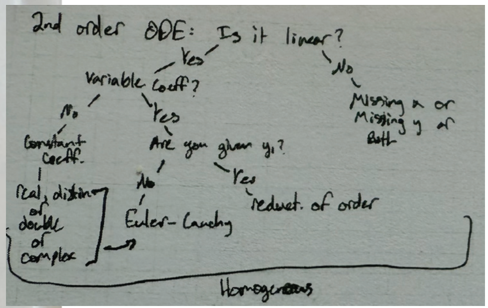
global error = RMS(local errors)

All **explicit** algorithms are **conditionally** stable and **implicit** is always (unconditionally) stable.

10. Laplace Transform

- Complete the squar if cannot separate denominator and use s-shift
- $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = F(s)$
- Solving ODEs: $ay'' + by' + cy = f \quad \to \quad Y(s) = \frac{F(s) + ay_0' + asy_0 + by_0}{as^2 + bs + c}$

Solving 2nd-Order ODEs



Second-order nonlinear ODE

$v = y'$ if no y → 1st-order ODE w/ $y'' = \frac{dv}{dx}$ easier

$v = y'$ if no x → $y'' = v\frac{dv}{dy}$ from $\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dy}\frac{dy}{dx}\frac{dy}{dx} = \frac{dv}{dy}v$

tip: evaluate C_1 w/ first IC before second integration

Second-order linear ODE $y'' + p(x)y' + q(x)y = 0.$

homogeneous y_h part still $y = y_h + y_p$

- basis of 2 unique solns $\{y_1, y_2\}$ → $y_h = c_1y_1 + c_2y_2$
  - lin indep if $y_2 = vy_1$ where v non-const
  - if y_1 & y_2 solns, $c_1y_1 + c_2y_2$ is soln by superposition principle
- method of reduction of order: if y_1 known $y_2 = vy_1 = \int \frac{e^{-\int p\,dx}}{y_1^2}dx\, y_1$ N.B.: no need for constant of integration
- if constant coeffs, characteristic eqn $a\lambda^2 + b\lambda + c = 0$ from ansatz $y = e^{\lambda t}$
  - 2 real λs: $y_h = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}$
  - double root λ: $y_h = c_1e^{\lambda t} + c_2te^{\lambda t}$ derived from method of reduction of order
  - complex λ: $y_h = Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x$ where $\alpha = \frac{-b}{2a}$, $\beta = \frac{\sqrt{4ac - b^2}}{2a}$
- if equidimensional $ax^2y'' + bxy' + cy = 0$ then Euler-Cauchy:
  indicial eqn $am^2 + (b-a)m + c = 0$ from ansatz $y = x^m$
  - 2 real ms: $y = c_1x^{m_1} + c_2x^{m_2}$
  - double root m: $y = c_1x^m + c_2x^m\ln|x|$
  - complex conj m: $y = Ax^{\alpha}\cos(\beta\ln|x|) + Bx^{\alpha}\sin(\beta\ln|x|)$ where $\alpha = \mathrm{Re}(m)$, $\beta = \mathrm{Im}(m)$

Method of Reduction of Order

Find y_2 from y_1 for $y_h = c_1y_1 + c_2y_2$

$y_2 = uy_1$, $y_2 = y_1\int \frac{e^{-\int p\,dx}}{y_1^2}dx$

homogeneous ODE if λ is double root: $y_h = c_1e^{-\frac{b}{2a}x} + c_2\mathbf{x}e^{-\frac{b}{2a}x}$

if λ complex, $y_h = Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x)$ where $\alpha = -\frac{b}{2a}$, $\beta = \frac{\sqrt{4ac - b^2}}{2a}$

Particular Part

Method of Variation of Parameters

Construct y_p from y_h's y_1 and y_2

$$y_p = -y_1\int \frac{y_2 r}{W}dx + y_2\int \frac{y_1 r}{W}dx$$

Wronskian $W = y_1y_2' - y_2y_1'$

Method of Underdetermined Coefficients

Table 8.1: Choices for y_p for undetermined coefficients method

If $r(x)$ is...	... then y_p is of the form...
C (a constant)	A
x^n (n must be a positive integer)	$A_nx^n + A_{n-1}x^{n-1} + ... + A_1x + A_0$
$e^{\gamma x}$ (γ either real or complex)	$Ae^{\gamma x}$
$\cos(\omega x)$ or $\sin(\omega x)$	$A\cos(\omega x) + B\sin(\omega x)$
$x^ne^{\gamma x}\cos(\omega x)$ or $x^ne^{\gamma x}\sin(\omega x)$	$(A_nx^n + A_{n-1}x^{n-1} + ... + A_1x + A_0)e^{\gamma x}\cos(\omega x) +$ $(B_nx^n + B_{n-1}x^{n-1} + ... + B_1x + B_0)e^{\gamma x}\sin(\omega x)$

Electrical & Mechanical Applications References

Electrical Applications

Electrical

capacitance C :: Farad

capacitor $I = \frac{dQ}{dt}$ $V_c = \frac{Q}{C}$

inductance L :: Henry

Inductor $V_L(t) = L\frac{dI}{dt}$

resistance R :: Ω

resistor $I = \frac{V}{R}$

Kirchoff's Current Law: current flowing into node = current flowing out from node

Voltage :: sum voltages in loop = 0

Existence & Uniqueness Thms

for $f = y'$, (x_0, y_0) in R_1 where R_1 is region where f continuous, R_2 region f_y continuous

if f continuous then 1+ solns ← existence thm

if f_y continuous then 1 soln ← uniqueness thm

springs

free oscillation – only I.C.s

forced oscillation – continuous external force acting on mass

damped – friction

undamped – no fricn

Free Undamped

– $A\sin(\omega t + \delta) = C_1\cos\omega t + C_2 \sin\omega t$ where $\delta = \text{atan2}(C_1, C_2)$

from $x'' + \frac{k}{m}x = 0$ where $\omega^2 = \frac{k}{m}$ $A = \sqrt{C_1^2 + C_2^2}$

$C_1 = A\sin\delta$ $C_2 = A\cos\delta$

– period $T = \frac{2\pi}{\omega}$, $f = \frac{1}{T}$, $\frac{\delta}{\omega}$ delay

Free damped

$x'' + 2\lambda x' + \omega^2 x = 0$

– viscous friction w/ air $\propto v$ so $mx'' = -kx - \beta x'$

– $r = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$

natural frequency $\omega^2 = \frac{k}{m}$

damping constant $\lambda = \frac{\beta}{2m}$

overdamped $\lambda > \omega$: two real roots r_1, r_2 both – so x exponentially decays

– friction/damping force $F_\beta = -\beta x'$

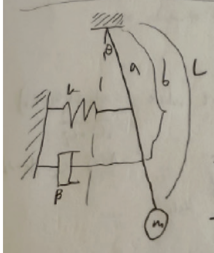
T=force * line ⊥ to line of force

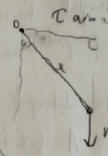
Rotational Oscillatory Systems

– $I\alpha = J\theta'' = \sum T = r_1F_1 + r_2F_2 + \ldots$

moment/torque arm – line ⊥ to force

– line of action – line along force's direction





T arm $= \ell\sin\theta$

$J\theta'' = T_k + T_\beta + T_m$

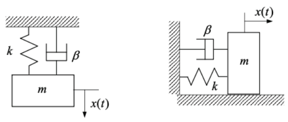
$T_k = -\underbrace{ka\sin\theta}_{F} \cdot \underbrace{a\cos\theta}_{r} = -ka^2\theta$

$T_\beta = \underbrace{-\beta\frac{d}{dt}(b\sin\theta)}_{F} \cdot \underbrace{b\cos\theta}_{r} = -\beta b^2\theta'$

$T_m = mg \cdot L\sin\theta = mgL\theta$

Setting up ODE for free oscillations – Find ω & λ

Translational oscillations – straight-line movement



- Use Newton's 2nd law for straight-line movement:

$$mx'' = \sum F = F_k + F_\beta$$

Where

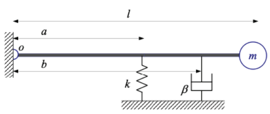
m = mass
$x(t)$ = displacement measured from equilibrium
F_k = spring force
F_β = friction (damping) force

- $F_k = -kx$ (a) proportional to distance x, (b) always carries a negative sign (−) (always oppose the motion); (c) if multiple springs, add up all individual spring forces
- $F_\beta = -\beta x'$ (a) proportional to velocity x', (b) always carries a negative sign (−) (always oppose the motion); (c) if multiple dampers, add up all individual friction forces
- Force due to the weight mg of the mass does not appear in the equation even in vertical oscillation since mg is canceled by the initial stretch/compression of the spring at rest.
- Put ODE in the form

$$x'' + \frac{\beta}{m}x' + \frac{k}{m}x = 0$$

to identify natural frequency $\omega^2 = k/m$ and damping constant $\lambda = \beta/(2m)$

Rotational oscillations – Horizontal bar



Notes:

- Use Newton's 2nd law for angular movement:

$$J\theta'' = \sum T = T_k + T_\beta$$

Where

J = moment of inertia of mass m about point of rotation
$\theta(t)$ = angular displacement measured from equilibrium
T_k = torque of spring force about point of rotation
T_β = torque of friction force about point of rotation

- Equilibrium is assumed to be horizontal
- Always assume small angle θ. Thus $\sin\theta \approx \theta$ and $\cos\theta \approx 1$
- $J = ml^2$, moment of inertia of m about point of rotation (provided or obtained from table)
- $T_k = -(F_k)(r_k) = -(a\sin\theta)(a\cos\theta) \approx -ka^2\theta$
  - ⋆ always carries a negative sign (−) (always oppose the motion)
  - ⋆ F_k is spring force
  - ⋆ r_k is torque arm = perpendicular distance from point of rotation to line of force F_k
- $T_\beta = -(F_\beta)(r_\beta) = -\beta\frac{d}{dt}(b\sin\theta)(b\cos\theta) \approx -\beta b^2\theta'$
  - ⋆ always carries a negative sign (−) (always oppose the motion)
  - ⋆ F_β is friction force
  - ⋆ r_β is torque arm = perpendicular distance from point of rotation to line of force F_β
- Torque due to the weight mg of the mass does not appear in the equation since it is canceled by the initial stretch/compression of the spring torque at rest.
- Put ODE in the form

$$\theta'' + \frac{\beta b^2}{ml^2}\theta' + \frac{ka^2}{ml^2}\theta = 0$$

to identify natural frequency $\omega^2 = \frac{ka^2}{ml^2}$ and damping constant $\lambda = \frac{\beta b^2}{2ml^2}$

System of ODEs

We now have a system of two 2nd-order ODEs

$$m_1x_1'' + \beta x_1' - \beta x_2' + (k_1 + k_2)x_1 - k_2x_2 = k_1Y_0\sin(\gamma t) \quad (9.56)$$
$$m_2x_2'' - \beta x_1' + \beta x_2' - k_2x_1 + (k_2 + k_3)x_2 = 0 \quad (9.57)$$

This system can be put in matrix form as shown below:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{Bmatrix} x_1'' \\ x_2'' \end{Bmatrix} + \begin{pmatrix} \beta & -\beta \\ -\beta & \beta \end{pmatrix}\begin{Bmatrix} x_1' \\ x_2' \end{Bmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} k_1Y_0\sin(\gamma t) \\ 0 \end{Bmatrix} \quad (9.58)$$

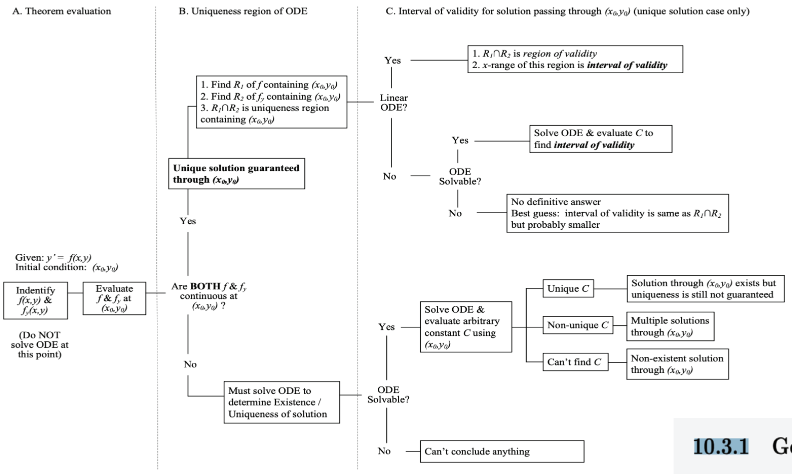
or

$$[\boldsymbol{M}]\{x''\} + [\boldsymbol{\beta}]\{x'\} + [\boldsymbol{K}]\{x\} = \{f\}, \quad (9.59)$$

where $[\boldsymbol{M}]$ is the mass matrix, $[\boldsymbol{\beta}]$ the friction coefficient matrix and $[\boldsymbol{K}]$ the *stiffness coefficient matrix*. It should be noted from Eq. (9.58) that, if the analysis is done correctly, the stiffness coefficient matrix should be symmetric.

Textbook & Code References

Existence & Uniqueness



$x = x_1 + x_2$

$F = k_1 x_1 \qquad F = k_2 x_2$

$x = x_1 + x_2 = \frac{F}{k_1} + \frac{F}{k_2} = F\left(\frac{1}{k_1} + \frac{1}{k_2}\right)$

$F = Kx$

$k = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} = \frac{k_1 k_2}{k_1 + k_2}$

ode45

options=odeset("RelTol", 1e-4, "AbsTol", 1e-6, "Refine", 12);

[t, y]=ode45(@myODE, tspan, y0, options);

$G(s) = \frac{s-1}{(s-1)^2 + 3^2}$

$H(s-1) = H(u) = \frac{u}{u^2+3^2} \rightarrow h(t) = \cos 3t$

$g(t) = e^t \cos 3t$

10.2.10 General procedure to perform Laplace transform

Given a function $f(t)$, below is the general procedure to find its Laplace transform $F(s)$ with the use of Table 10.1. Note: Laplace transform of a product of two functions, e.g., $f(t)g(t)$, is beyond the scope of this class, except two special products $e^{at}f(t)$ and $t^n f(t)$.

- Break down $f(t)$ into components that match functions in the t-domain column, e.g., $\cos(\omega t)$, $\sin(\omega t)$, etc.
- Find the corresponding transform $F(s)$ in the s-domain column
- If the function in t-domain is of the form $e^{at}f(t)$, then
  - ⋆ leave out e^{at}
  - ⋆ find Laplace transform of $f(t)$ to get $F(s)$
  - ⋆ apply s-shift (transform pair #12)
- If the function in t-domain is of the form $t^n f(t)$, then
  - ⋆ leave out t^n
  - ⋆ find Laplace transform of $f(t)$ to get $F(s)$
  - ⋆ apply transform pair #9
- If the function in t-domain is of the form $u(t-a)f(t)$, then use the procedure for transforming a "cut-off function" (transform pair #13)

10.3.1 General procedure for inverse Laplace transform

Given a function in s-domain, $F(s)$, below is the general steps to find its inverse Laplace transform $f(t)$ using Table 10.1.

- Use partial fractions to bread down $F(s)$ into components that match functions in the s-domain column, e.g., $\frac{1}{s-a}$, $\frac{s}{s^2+\omega^2}$, etc.
- Find the corresponding inverse transform function $f(t)$ in the t-domain column
- If the function in s-domain is of the form $e^{-as}F(s)$, then
  - ⋆ leave out e^{-as}
  - ⋆ find inverse Laplace transform of $F(s)$ to get $f(t)$
  - ⋆ apply t-shift (transform pair #13)
- If the function in s-domain is of the form $F(s-a)$, then
  - ⋆ find inverse Laplace transform of $F(s)$ to get $f(t)$
  - ⋆ multiply $f(t)$ by e^{at} in t-domain (using transform pair #12)

Forward Euler

```
function [t,y] = euler(f,t0,tf,y0,h)
        t = t0:h:tf;
        y(1)=y0;
        for n = 1:length(t)−1
                y(n+1) = y(n) + h*f(t(n),y(n));
        end
end
```

Backward Euler

```
function [t, y]=euler(f, t0, tf, y0, h)
        t=t0:h:tf;
        y(1)=y0;
        for i=1:length(t)-1
                y(i+1)=... %derive eqn first
        end
end
```

```
function [t, y]=backward_euler(t0, tf, h, y0)
    t=t0:h:tf;
    y(1)=y0;
    for n=1:length(t)-1
        y(n+1)=(y(n)+4*h*cos(3*t(n+1)))/(h+1);
    end
end
```

```
NUM_POINTS=20;
x_points=linspace(-1, 1, NUM_POINTS);
y_points=linspace(-2, 2, NUM_POINTS);

[X, Y]=meshgrid(x_points, y_points);

v_x=ones(NUM_POINTS, NUM_POINTS);
v_y=X+Y.*(1-Y); %ODE(X, Y)

% Normalize
length = sqrt(v_x.^2+v_y.^2);
v_x = v_x./length;
v_y = v_y./length;

quiver(X, Y, v_x, v_y);
title("Direction Field for y'=x+y(1-y)")
xlabel("X")
ylabel("Y")
axis([-1, 1, -2, 2])
```

```
function [t, y]=euler(ode, t0, tf, h, y0)
    t=t0:h:tf;
    y(1)=y0;
    for i=1:length(t)-1
        y(i+1)=y(i)+h*ode(t(i), y(i));
    end
end
```

```
function res=rms(lst)
    res=norm(lst)/sqrt(length(lst));
end
```

```
function [t, y]=rk4(f, t_bounds, y0, h)
    t=t_bounds(1):h:t_bounds(2);
    y(1)=y0;
    for i=1:length(t)-1
        k1=f(t(i), y(i));
        k2=f(t(i)+h/2, y(i)+h/2*k1);
        k3=f(t(i)+h/2, y(i)+h/2*k2);
        k4=f(t(i)+h, y(i)+h*k3);
        y(i+1)=y(i)+h*(k1/6+k2/3+k3/3+k4/6);
    end
end
```

```
sigma=10; b=8/3; r=20;
tspan=[0,30];
Y0=[1,1,1];

[t, Y]=ode45(@(t,y) lorenz(t, y, sigma, r, b), tspan, Y0);

plot(t, Y)
xlabel('t')
ylabel('x, y, z')
legend('x(t)', 'y(t)', 'z(t)')
title('Lorenz Attractor, x(t), y(t), z(t)')

function Yp=lorenz(t,Y,sigma,r,b)
    Yp=zeros(3,1);
    Yp(1)=sigma*(Y(2)-Y(1));
    Yp(2)=r*Y(1)-Y(2)-Y(1)*Y(3);
    Yp(3)=Y(1)*Y(2)-b*Y(3);
end
```

Sum and Difference Formulas

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$$

Sum to Product Formulas

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin(\alpha) - \sin(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Cofunction Formulas

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta) \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec(\theta) \qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc(\theta)$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta) \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan(\theta)$$

Double Angle Formulas

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

Half Angle Formulas

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$$

Half Angle Formulas (alternate form)

$$\sin^2(\theta) = \tfrac{1}{2}(1 - \cos(2\theta)) \qquad \tan^2(\theta) = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

$$\cos^2(\theta) = \tfrac{1}{2}(1 + \cos(2\theta))$$